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TIME MINIMIZATION DELIVERY PLANNING WITH THE TIME-QUANTITY DEPENDENCE

Summary. One of the classical problems in transportation planning is represented minimizing the maximal delivery time of a uniform commodity between sources and destinations, known as the Bottleneck Transportation Problem (BTP). It assumes that a fixed transportation time – independent of the quantity of the transported commodity – is assigned to each source-to-destination route. In some cases, however, the quantity of the transported commodity may affect the transportation time, e.g., because of the duration of loading/unloading the commodity to/from the vehicle. Extensions of the BTP as well as the closely related Total Time Minimization Transportation Problem (TTMTP) which include the linear time-quantity dependence of the delivery time are considered. Whereas similar optimization problems known in the literature are nonlinear, linear programming is used in this research. Linear optimization provides better performance of the optimization software in comparison with nonlinear optimization. The above fact is illustrated by improving solutions to the problems known in the literature. A detailed insight into the issue of the existence of integer optimal solutions and interpretations of optimal solutions is also provided.

Keywords: transport optimization, bottleneck transportation problem, total time minimizing transportation problem, integer linear programming

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1. INTRODUCTION AND LITERATURE REVIEW

Transportation is one of the most important human activities, essential for businesses, public services, emergency aid, and military operations. The optimization of transportation operations has always been an interest of decision makers. However, the mathematical complexity of many real-world problems is often an obstacle in the efficient search for optimal solutions. The beginnings of the contemporary scientific approach to transportation optimization date back to the second quarter of the 20th century, when papers by Tolstoi [1], Kantorovich [2] and Hitchcock [3] were published. Especially the problem considered by Hitchcock, known as the Hitchcock Transportation Problem, the Standard Transportation problem (STP), the Cost Minimization Transportation Problem (CMTP) or just the Transportation Problem (TP) was a "starting point" in the modern optimization of transport operations. It can be briefly described as the allocation of a uniform commodity on the routes connecting sources with limited supplies to destinations with defined demands. The goal of the allocation is to minimize the total transportation (also referred to as shipping) cost while the demands are satisfied, and the limits of supplies are not exceeded. Moreover, the transportation cost on each route is the product of a fixed unit cost and the amount of the commodity and the total transportation cost is the sum of the costs on all routes, making the total transportation cost a multivariable linear function. In 1951 Dantzig [4] expressed the STP as a linear programming problem and solved it using the simplex method. Because of a specific mathematical form of the STP which allows for the use of computational techniques not available for "general" linear programming problems, many various solving methods like the Stepping Stone Method [5], Vogel's Approximation Method or VAM [6], Least Cost Method or LCM [7], modified Stepping Stone Method [8], Lowest Allocation Method or LAM [9], some modifications of Vogel's Method [10-11] have been developed.

Even though TP was itself a great achievement, it turned out as early as in 1950's that reliable modeling the decisions related to the real-world transportation required more elaborate mathematics, which resulted in creating many extensions of STP (see e.g. [13]). In particular, minimization of the total cost (of the transportation itself only or increased by some other costs) was no longer a unique criterion of optimality used in extensions of STP.

Among the optimality criteria other than the minimization cost, the time minimization is one of the most important ones. This criterion requires a more detailed description in the context of optimization of transportation activities because it has two different meanings under the same name. The first meaning is "to minimize the time in which the commodity is delivered to all the destinations, assuming that the transportation started simultaneously at all the sources". This criterion is modeled under the names the Bottleneck/Time Minimizing Transportation Problem. Another meaning can be "to minimize the total time in which the transportation of the commodity is performed (the sum of the transportation times on all routes connecting sources and destinations)". The latter criterion is modeled as the Total Time Minimizing Transportation Problem.

The both meanings of "the time minimization" refer to two separate time-related optimality criteria that occur instead of the cost minimization criterion of STP (or some of its extensions) while the constraints regarding the flow of the commodity remain unchanged. The first criterion results from situations in which the time of completing the deliveries, not the cost, matters, like deliveries of emergency supplies to locations affected by disasters, ammunition to battlefields, perishable goods to customers, etc. The second one results from the necessity of minimizing the usage of scarce resources like the work time of drivers or plane/ship crews. To continue, an important assumption must be made. Namely, for both of the abovementioned criteria, it is assumed that there is a fixed transportation time assigned to each source-destination pair, which does not depend on the quantity of the transported commodity. A motivation for such an assumption is that if a non-zero flow of the commodity is scheduled to a route, then the commodity on this route is transported in a single vehicle or in a "team" of vehicles traveling together (e.g., a single truck or a convoy of trucks). For simplicity reasons, in later considerations we assume that if a positive quantity of the commodity is transported, then a single vehicle is always used for transportation on any source-to-destination route (and such a route is then called "a used route"). When the time-optimization criteria are applied, the time of the transportation on any specific route is a discontinuous function of the quantity of the commodity, which equals zero for the argument equal to zero and equals a constant positive number for a positive argument.

The difference between the two abovementioned criteria is the objective function. For the first one, it is the maximum of the transportation times over all the routes, whereas for the second one, it is the sum of the transportation times over all the routes. The first case is more represented in the literature, and it is known as the Bottleneck Transportation Problem (BTP) or the Time Minimizing Transportation Problem (TMTP). The second problem is known as the Total Time Minimizing Transportation Problem (TTMTP).

The earliest time-optimal version of the problem known today as BTP or TMTP was formulated by Barsov in 1959 [15], who named it the Transportation Problem Time with a Time Criterion. Further developments in solving BTP are due to Nesterov [16], Grabowski [17,18] (as the Transportation Problems with Minimal Time), Szwarc [19,20] (as the Time Transportation Problem) and Hammer (as the Time Minimizing Transportation Problem) [21]. All those authors used modifications of the simplex method to solve the problem. Garfinkel and Rao in [22] used an approach based on the Hungarian method instead. Garfinkel and Rao were also the first ones to use the name "Bottleneck Transportation Problem". More progress was achieved by Sharma and Swarup [23], Bhatia, Swarup and Puri [24], Seshan and Tikekar [25], Issermann [26]. Another approach to solving BTP is to consider it as a special case of so-called bottleneck linear programming [27].

BTP allowed to model decisions that were impossible to be handled by STP. BTP in its "pure" form, however, also turned out not to be sufficient for modeling some real-world decisions, and this is why many extensions of BTP have been created in response to the needs of decision-makers. Those extensions included the Capacitated Bottleneck Facility Location Problem [28], the Bottleneck Capacitated Transportation Problem with Bounds on Rim Conditions [29], the Time Minimizing Transportation Problem with Mixed Constraints [30].

The time optimization in transportation planning does not need to occur instead of the cost optimization as a unique criterion of optimality. They may "coexist", being both – not necessarily equally – important to a decision maker. This is why some models that consider both optimality criteria have also been created. In [31] the Bottleneck-Cost Transportation Problem was introduced as a bi-criteria optimization problem in which the total cost and the delivery time are minimized simultaneously. In [32] two complementary bi-level programming models are considered in which either minimizing the maximal delivery time is the primary and minimizing the cost is the second-level optimality criterion or vice versa. In [33] the Constrained Bottleneck Transportation Problem (CBTP) was introduced, in which a budget constraint (the maximal total transportation cost) was added.

Dropping the assumption of the lack of dependence of the transportation time on the actual quantity of the transported commodity, which is a main issue investigated in this paper, has also been considered but to a limited extent only. In [34] such a dependence was introduced as an increasing piecewise constant function. This dependence resulted from the necessity of performing multiple trips for each source-destination pair due to insufficient capacities of the available vehicles. A version of BTP with the transportation time being proportional to the quantity of the delivered commodity instead of the fixed time was considered in [33]. The quantity-dependent and fixed components of the delivery time were joined together into one model in [35] and [36]. In those papers, the unloading time proportional to the amount of the delivered commodity, added to the fixed time was considered. In [36] the standard BTP model was extended by including both the loading and unloading time and also by considering using many vehicles per route because of the limited availability of vehicles. In this paper the concepts of the time-quantity dependence from [35], [36] and [36] are joined together and extended into one mathematical model (without multiple vehicle usage considered in [36], however).

This paper is intended to present a comprehensive theoretical background and to provide an efficient computational method for the transportation problem with the time minimization and the time-quantity dependence. Whereas this paper is devoted mainly to an extension of BTP that includes the time-quantity dependence, an analogical extension of TTMTP is also considered. Particular goals of the paper are the following.

The first goal is to formulate an extension of BTP to the case in which the minimal delivery time depends also on the quantity of the transported commodity, based on the research in the field. This extension concerns a wider range of practical applications of the model than those defined in [35] and [36], but without changing its mathematical form. A new name for the considered problem is also introduced to better conform to the existing naming conventions in the field. An analogical extension of TTMTP (based on [38]) is also introduced.

The second and most important goal is to reformulate the proposed extension of BTP which is, like the "pure" BTP, a nonlinear optimization problem, to be a linear optimization problem. The reason for the reformulation is practical. Whereas the linear formulation of the considered problem is formally equivalent to the nonlinear one, the performance of optimization software may be poor in the case of nonlinear optimization, resulting in calculating worse, suboptimal solutions instead of optimal ones. This phenomenon occurred in [36]. Unfortunately, no efficient method of obtaining optimal solutions was presented or even suggested. In this paper it was shown that optimizing a linear version of the considered problem allows finding an optimal solution. An analogical linearization of the extension of TTMTP is also performed.

The third goal is to discuss the existence of integer-valued optimal solutions of the considered optimization problems depending on the values of the parameters, as well as the necessity of imposing integer constraints on the variables.

The paper has the following structure. Section 2 reviews BTP and TTMTP as formulated nonlinear programming problems and presents their extensions, which include a component of the delivery time depending on the quantity of the transported commodity. Those extensions, primarily nonlinear, are later reformulated as linear programming problems. The issues related to integer-valued optimal solutions are also discussed. In Section 3, two example problems, one from [35] and one from [36], are solved as linear programming problems to show the improvement in the quality of the solutions obtained thanks to using linear optimization instead of nonlinear. The results of other calculations related to the existence of integer optimal solutions. Section 5 contains a discussion on the considered models themselves as well as on the results of the performed calculations. In Section 6, final conclusions are presented.

2. DATA AND METHODS

The main topic of this section is an extension of BTP with the quantity-time dependence and its linearization. An analogical extension and linearization will be performed on TTMTP.

Let us introduce the necessary notation, common to BTP and TTMTP, and their extensions. A uniform commodity is to be delivered from m sources to n destinations. The transportation time for each "source-destination" pair is constant and does not depend on the amount of the commodity. However, if no commodity is transported, then the transportation time is obviously equal to zero. Each "source-destination" pair is called a *route* from source i to destination j. The following parameters are given:

- a_i – maximal possible supply from source i $(i = 1, \dots, n)$;
- b_j – demand of destination j $(j = 1, \dots, m)$;
- t_{ij} – transportation time of any non-zero quantity of the commodity from source i to destination j $(i = 1, \dots, n;\ j = 1, \dots, m)$;
- d_{ij} – the maximal capacity of the route from source i to destination j (optional parameters) $(i = 1, \dots, n;\ j = 1, \dots, m)$.

The variables denote quantities of the commodity through all the routes:

- x_{ij} – amount of the commodity transported from source i to destination j $(i = 1, \dots, n;\ j = 1, \dots, m)$.

The above parameters and variables are identical to those in the STP except for t_{ij}. On the other hand, no cost-related parameters occur in BTP and TTMTP. The interpretation of t_{ij} can be just as simple as "the trip time from i to j". An assumption is also made that the delivery on each route is performed by a single vehicle. The value t_{ij} includes the "true" time in motion (the driving time), but it can also include time required for the rest or refueling or any other inevitable breaks.

Finally, the delivery time of any (zero or positive) quantity of the commodity from source i to destination j can be then expressed as the following conditional formula:

$$T_{ij}(x_{ij}) = \begin{cases} 0 & x_{ij} = 0 \\ t_{ij} & x_{ij} > 0 \end{cases} \quad (i = 1, \dots, n;\ j = 1, \dots, m). \tag{1}$$

The goal of transportation planning in the BTP model is to complete the deliveries of the commodity as soon as possible, assuming that they start simultaneously from all the sources. In other words, the latest time of completing all the deliveries is calculated. Alternatively, the goal can be formulated as all the deliveries must reach their destinations simultaneously, and the earliest start time of some deliveries is calculated. This goal means that, no matter if for the latest time of completing or the earliest start time, the maximum of $T_{ij}(x_{ij})$ over all x_{ij} must be minimized. Finally, the objective function of BTP is the following:

$$\max_{x_{ij} \geq 0} \{T_{ij}(x_{ij})\} \to \min \quad (i = 1, \dots, n;\ j = 1, \dots, m). \tag{2}$$

The only difference between TTMTP and BTP is the objective function, which in TTMTP is just the sum of the transportation times on all the routes:

$$\sum_{i=1}^{m}\sum_{j=1}^{n} T_{ij}(x_{ij}) \rightarrow \text{min.} \tag{3}$$

Both BTP and TTMTP share the same set of constraints, initially introduced in STP. For simplicity of the notation, we also assume additionally that the problem under consideration is balanced, i.e., the sum of all the maximal possible supplies (the total maximal possible supply) is equal to the sum of all the demands (the total demand):

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j, \tag{4}$$

Both objective functions (2) and (3) are subject to the constraints:

$$\sum_{j=1}^{n} x_{ij} = a_i \ \ (i = 1, \dots, m) \tag{5}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ \ (j = 1, \dots, n) \tag{6}$$

$$x_{ij} \geq 0 \ \ (i = 1, \dots, n;\ j = 1, \dots, m) \tag{7}$$

$$x_{ij} \leq \ d_{ij} \ \ (i = 1, \dots, n;\ j = 1, \dots, m) - \text{optional.} \tag{8}$$

Constraints (8) are imposed only if it is necessary, i.e., if there are upper bounds on the capacities of the routes. The details of using (8) will be discussed later.

A remark on integer solutions is necessary. A well-known property of STP concerning the existence of integer-valued optimal solutions is now reminded. If all the parameters a_i $(i = 1, \dots, n)$, b_j $(j = 1, \dots, m)$ and, if defined, d_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$ are integer, then the constraints (5-8) define a feasible set in which that all the basic feasible solutions (the coordinates of the corner points of the feasible set) are all integer. In this case, the optimal solutions (one or more) are also integer because they are found among the basic feasible solutions. This property is not restricted to STP, however, it holds also for BTP and TTMTP, as explained below.

In [39] it was proved that the objective function of BTP (2) is a concave function, and due to the concavity of the objective function, the search for an optimal solution is restricted to the set of the basic feasible solutions only. Thus, if a_i, b_j and, if defined, d_{ij} in BTP are all integer, then the basic feasible solutions, and what follows, also optimal solutions are integer. This is why there is no need to impose integer constraints on x_{ij}.

As to TTMTP, it is mathematically equivalent to the Pure Fixed Charge Transportation Problem (PFCTP) [40] in which the fixed cost parameters are replaced with the time parameters t_{ij}. PFCTP itself is a special case of the Fixed Charge Transportation Problem (FCTP) [41], [42]. One of the basic properties of all the linear fixed charge problems is that the optimum is attained at a corner point of the feasible set of the continuous variables x_{ij} [42]. So again, if a_i, b_j and, if defined, d_{ij} in TTMTP are all integers, then the basic feasible solutions and optimal solutions are also integer.

Both BTP and TTMTP, even though useful in many real-world applications, may be, in some circumstances, too simple to meet the needs of decision makers. In particular, they do not take into account the fact that the delivery process on any route may depend on the quantity of the commodity (obviously besides the dependence "zero time for no commodity/fixed time for a positive quantity of the commodity"). As it previously mentioned, this dependence was considered in [35] and [36] as the unloading time and in [36] as both the loading and unloading times. In all the above cases, loading and unloading times were assumed to be proportional to the quantity of the delivered commodity on each route. However, it is easy to notice that the abovementioned dependencies can be interpreted in a wider sense than that formulated in [35], [36] and [36]. Namely, it is not loading and unloading times only that sometimes must be included in the total delivery time. There can also be a possible increase in the transportation time itself caused by slowing down the vehicle carrying the heavy load. The sum of the loading time, the unloading time, and the transportation time increase, all of which depend on the quantity of the transported commodity being transported, will be referred to as the *quantity-dependent time*. The quantity-dependent time added to the quantity transportation time t_{ij} gives the value of the actual delivery time. The simplest way to include in an optimization process the quantity-time dependence is to assume that the quantity-dependent time is proportional to the quantity of the commodity. Obviously, loading and unloading times of the same amount of commodity do not need to be equal. They may also differ at various locations due to the availability of necessary equipment and staff. The quantity-dependent transportation time increase may also be different on various routes, for example, because of various road quality.

Extensions of BTP and TTMTP that include the quantity-dependent time into the optimization criteria as described above will be named the Quantity Dependent Bottleneck Transportation Problem (QDBTP) and the Quantity Dependent Total Time Minimizing Transportation Problem (QDTTMTP), respectively.

We need to introduce the following additional parameters in order to extend the previously considered models to QDBTP and QDTTMTP, respectively:

- t'_{ij} – the unit quantity-dependent time, i.e., the sum of the times of loading and unloading one unit of the commodity on the route from source i to destination j plus the increase of the transportation time resulting from transporting one unit of the commodity on the route from source i to destination j $(i = 1, \dots, n,\ j = 1, \dots, m)$.

The assumption that the quantity-dependent time is proportional to the quantity of the commodity means that the quantity-dependent time for a route connecting i to j is $t'_{ij}\, x_{ij}$ and the delivery time for that route is $T_{ij}(x_{ij}) + t'_{ij}\, x_{ij}$. The parameters t'_{ij} are used in the objective functions of QDBTP and QDTTMTP only. However, they all share the same set of constraints, initially introduced in STP (with one exception connected with integer constraints in QDBTP, discussed later).

The objective function for QDBTP is:

$$\max_{i,j} \{T_{ij}(x_{ij}) + t'_{ij}\, x_{ij}\} \to \min \ (i = 1, \dots, n;\ j = 1, \dots, m). \tag{9}$$

The objective function for QDTTMTP:

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\left(T_{ij}\left(x_{ij}\right)+t'_{ij}\,x_{ij}\right)\rightarrow \min. \tag{10}$$

Both (9) and (10) are subject to the constraints (5-8).

Whereas the constraints (5-8) are linear, both objective functions (9) and (10) are not linear and not smooth (see Figure 2). The lack of linearity and smoothness of the objective functions (9) and (10) results from nonlinearity of $T_{ij}\left(x_{ij}\right)$, $(i=1,\dots,n;\ j=1,\dots,m)$. Moreover in (9), nonlinear formulas $T_{ij}\left(x_{ij}\right)+t'_{ij}\,x_{ij}$ are arguments for another nonlinear function max.

The calculations in [36] showed that optimizing the QDBTP models with the objective function (9) (transformed, but still nonlinear) may lead to ambiguous, suboptimal solutions. Instead, in this paper, a linearization, i.e., a transformation of QDBTP into a linear programming problem, is considered. Linearization of QDTTMTP is performed by using the same technique.

Linearization described below consists of well-known transformations, applied to the two considered objective functions.

The first step is linearization of $T_{ij}\left(x_{ij}\right)$. Whereas explicit constraints (8) restricting the route capacities in STP and, what follows, in BTP/QDBTP and TTMTP/ QDTTMTP are optional, it does not mean that x_{ij} can attain arbitrarily large values. Instead, implicit capacity constraints for all the routes exist. Indeed, by (5) and (6), the quantity of the transported commodity on each route cannot exceed the minimum of the maximal possible supply at the source and the demand at the destination:

$$x_{ij}\leq M_{ij}=\min\left\{a_i,b_j\right\}\quad (i=1,\dots,n;\ j=1,\dots,m). \tag{11}$$

The value M_{ij} is an implicit upper bound on the commodity flow on the route connecting i to j. An optional maximal route capacity d_{ij} may affect the feasible set and an optimal solution only if $d_{ij}<M_{ij}$. From a practical point of view, in BTP/QDBTP and TTMTP/ QDTTMTP, parameters d_{ij} must be specified explicitly, usually if the vehicle can carry the maximal payload not larger than M_{ij} (e.g., due to the technical specification of the vehicle or restrictions imposed by the condition of the road infrastructure). Let M denote the maximal common upper bound on the commodity flow for all routes, which is the maximum of all the upper bounds M_{ij}:

$$M=\max_{i,j}M_{ij}=\max_{i,j}\min\left\{a_i,b_j\right\}=\min\left\{\max a_i\,,\max b_j\right\}\quad (i=1,\dots,n;\ j=1,\dots,m). \tag{12}$$

By (11) and (12):

$$x_{ij}\leq M\ (i=1,\dots,n;\ j=1,\dots,m) \tag{13}$$

The inequalities (13) are satisfied for any x_{ij}, so adding them as constraints does not affect the feasible set. After a slight transformation, they are used to linearize $T_{ij}\left(x_{ij}\right)$.

Next, new auxiliary binary variables are introduced:

- y_{ij} – indicating if a non-zero quantity of the commodity is transported along the route from i to j ($y_{ij}=1$ – “the route from i to j is used”/”one vehicle is travelling from i to j”) or not ($y_{ij}=0$ – “the route from i to j is not used”/”zero vehicles are travelling from i to j”). $(i=1,\dots,n;\ j=1,\dots,m)$.

Let us notice that (1) can be expressed as:

$$T_{ij}(x_{ij}) = \begin{cases} 0 = t_{ij} \cdot 0 & x_{ij} = 0 \\ t_{ij} = t_{ij} \cdot 1 & x_{ij} > 0 \end{cases} \quad (i = 1, \dots, n;\ j = 1, \dots, m). \tag{14}$$

By the definition of y_{ij}, (14) can be expressed as:

$$T_{ij}(x_{ij}) = t_{ij} y_{ij} \quad (i = 1, \dots, n;\ j = 1, \dots, m). \tag{15}$$

The following inequalities define the relationships between x_{ij} and y_{ij} which provide an indication of whether a non-zero amount of commodity is transported ($x_{ij} > 0$) or no commodity ($x_{ij} = 0$) is transported:

$$x_{ij} \le M y_{ij} \ (i = 1, \dots, n;\ j = 1, \dots, m). \tag{16}$$

Inequalities (16) will become another group of constraints of the models, added to (5-8). Obviously, if $y_{ij} = 1$, then (16) is equivalent to (13) and it does not affect the feasible set. If $y_{ij} = 0$, then also $x_{ij} \le 0$, what, together with $x_{ij} \ge 0$ (7) results in $x_{ij} = 0$.

The transformation (15) is sufficient to linearize QDTTMTP by replacing (1) with (15) in (3). However, this is not the case for QDBTP as another transformation of the optimization model must be done because of the presence of the max function.

The objective function of QDBTP (9) can be expressed now in the form:

$$\max_{i,j} \{T_{ij}(x_{ij}) + t'_{ij}\, x_{ij}\} = \max_{i,j} \{t_{ij} y_{ij} + t'_{ij}\, x_{ij}\} \quad (i = 1, \dots, n;\ j = 1, \dots, m). \tag{17}$$

In order to “remove” the max function from (17), a new variable z is defined:

$$z = \max_{i,j}\{t_{ij} y_{ij} + t'_{ij}\, x_{ij}\} \quad (i = 1, \dots, n;\ j = 1, \dots, m). \tag{18}$$

Since

$$t_{ij} y_{ij} + t'_{ij}\, x_{ij} \le \max_{i,j}\{t_{ij} y_{ij} + t'_{ij}\, x_{ij}\} \quad (i = 1, \dots, n;\ j = 1, \dots, m) \tag{19}$$

then by (18) and (19):

$$t_{ij} y_{ij} + t'_{ij} x_{ij} \le z \quad (i = 1, \dots, n;\ j = 1, \dots, m). \tag{20}$$

If (20) is added as another group of constraints to (5-8) and (16), then minimizing a new objective function composed of a single variable z is equivalent to minimizing the original objective function of QDBTP (3). The variable z and constraints (20) are obviously redundant in QDTTMTP.

Finally, the linearized versions of QDBTP and QDTTMTP are:

$$z \to \min \quad \text{(QDBTP)} \tag{21}$$

$$\sum_{i=1}^{m} \sum_{j=1}^{n} (t_{ij} y_{ij} + t'_{ij}\, x_{ij}) \to \min \quad \text{(QDTTMTP)} \tag{22}$$

The subject to the constraints (common for QDBTP and QDTTMTP, unless specified):

$$\sum_{j=1}^{n} x_{ij} = a_i \ \ (i = 1, \dots, m) \tag{23}$$

$$\sum_{i=1}^{m} x_{ij} = b_j \ \ (j = 1, \dots, n) \tag{24}$$

$$x_{ij} \geq 0 \ \ (i = 1, \dots, n; \ j = 1, \dots, m) \tag{25}$$

$$x_{ij} \leq \ d_{ij} \ \ (i = 1, \dots, n; \ j = 1, \dots, m) \text{ – optional} \tag{26}$$

$$y_{ij} \text{ binary } (i = 1, \dots, n; \ j = 1, \dots, m) \tag{27}$$

$$x_{ij} \leq My_{ij} \ (i = 1, \dots, n; \ j = 1, \dots, m) \tag{28}$$

$$t_{ij}y_{ij} + t'_{ij}x_{ij} \leq z \ \ (i = 1, \dots, n; \ j = 1, \dots, m) \text{ (QDBTP only)} \tag{29}$$

$$x_{ij} \text{ integer } (i = 1, \dots, n; \ j = 1, \dots, m) \text{ (if necessary, QDBTP only).} \tag{30}$$

In order to provide consistent numbering for linear versions of QDBTP and QDTTMTP, some numbers of the formulas are repeated, where (23-26) are identical to (5-8), (28) to (16) and (29) to (20) respectively.

Obviously, if all the $t'_{ij} = 0$, then QDBTP and QDTTMTP – both as nonlinear and linear optimization models – reduce to BTP and TTMTP, respectively.

The linear formulation of BTP (21), (23-29) is almost identical to CBTP Model II in [33]. The only difference is the lack of the budget (the maximal transportation cost) constraint, present in [33]. On the other hand, QDBTP in its linear version can be considered as an extension of CBTP Model II from [33] which includes the quantity-dependent time delivery time, and with the budget constraint being removed.

Introducing the constraints (30) needs some more explanation. As it has been stated, if a_i, b_j and, if defined, d_{ij} in BTP or TTMTP are all integer, then the basic feasible solutions, and what follows, also optimal solutions are integer. This property also holds for QDTTMTP which is mathematically equivalent to FCTP, where the fixed cost parameters are replaced with the time-parameters t'_{ij} and the variable cost parameters are replaced with the time-parameters t_{ij}. Then, by [42], the existence of integer optimal solutions is "guaranteed" in QDTTMTP like in TTMTP. However, unlike in case of BTP, in QDBTP integer constraints on x_{ij} must be imposed, if necessary. Solutions of example problems later in this paper showed that without imposing integer constraints on the variables x_{ij} integer optimal values of x_{ij} in QDBTP are not "guaranteed" even if the parameters if a_i, b_j and, if defined, d_{ij} are all integers.

Before introducing further theoretical considerations and presenting the results of calculations for example problems, an important remark on the variables y_{ij} must be made. The variables y_{ij} are auxiliary binary variables that are defined as indicators of a non-zero ($y_{ij} = 1$) or zero ($y_{ij} = 0$) flow of the commodity from source i to destination j. However, because of properties resulting from (28), the interpretation of the value $y_{ij} = 1$ as an indicator of a non-zero flow from source i to destination j is to some extent ambiguous. Namely, if $y_{ij} = 0$, then $x_{ij} = 0$ what means that the value $y_{ij} = 0$ indicates zero flow of the commodity from i to j, but not necessarily each zero flows must be "marked" by $y_{ij} = 0$. If $y_{ij} = 1$, then, by (28), there can be either $x_{ij} > 0$ or $x_{ij} = 0$ what means that the value $y_{ij} = 1$ indicates all the non-zero flows of the commodity, but it may also indicate zero flows on some routes. This ambiguity concerning the interpretation of $y_{ij} = 1$ does not affect finding an optimal solution – neither of QDBTP nor of QDTTMTP – understood as all the values x_{ij} for which the objective function is minimized. However, in QDBTP it is possible that for some optimal values $x_{ij} = 0$, the corresponding optimal values $y_{ij} = 1$. It is necessary to know the above fact in order to understand correctly the results of the calculations. On the other hand, a couple of values $y_{ij} = 1$ and $x_{ij} = 0$ can never happen in an optimal solution of QDTTMTP (it is possible only in the case of nonoptimal feasible solutions).

In real-world applications, BTP/QDBTP and TTMTP/QDTTMTP do not need to be balanced problems. If the total maximal possible supply of sources is not equal to the total demand of destinations, i.e., (4) does not hold, or even more sophisticated so-called mixed constraints occur, then (5) or (6) must be replaced with appropriate sets of constraints.

3. RESULTS

A linear programming formulation of QDBTP was verified by addressing the problems known from the literature of the subject, namely from [35] and [36]. The problem considered in [36] concerns some extension of QDBTP with the opportunity of using multiple vehicles per route, and this is why it was not fully suitable for comparative calculations, as the models with one vehicle per route only are considered in this paper. Both [35] and [36] consider the unloading time only as the quantity-dependent component of the delivery time. However, since the definition of QDBTP joins together all the delays added to the "standard" transportation time on any route as the quantity-dependent time, the abovementioned examples are obviously instances of QDBTP. Below there are more precise characteristics of the examples and the solutions obtained in the quoted papers compared to the solutions calculated for this paper. All the solutions calculated for this paper were obtained by using linear programming with binary variables y_{ij} and, when necessary, integer constraints imposed on variables x_{ij}. Detailed references to the formulas for each model are given below the tables with the results (also for quoted results of nonlinear calculations) [44].

As to the optimization software, Excel 365 Version 2406 and the add-in OpenSolver 2.9.4 [43] running on Windows 11 PRO 23H2 were used. The computer was a Dell Latitude 5440 with a 13th Gen Intel Core i5-1345U 1.60 GHz processor, 16 GB of RAM, and NVMe SN740 WD 1 TB SSD.

Whereas a commonly used symbolic notation for optimal solutions of optimization problems is to use the symbols of the variables with the asterisk in the superscript, in this section the meaning of this notation will be widened. The symbols x^*_{ij} and y^*_{ij} will stand for a solution calculated by the optimization software, no matter whether it is optimal or suboptimal.

Analogically, in this section t_{min} stands for the minimal delivery time calculated by the optimization software. This time may be either optimal or suboptimal.

Whereas the problems both from [35] and [36] were solved, the results for the problem from [36] only are presented below. The reason is that the problem from [35] turned out to have the same delivery time when solved as a linear programming problem. Instead, in the case of the problem from [36] using linear optimization resulted in a significant improvement of the solution.

Example

Problem from [36]: 9 sources, 16 destinations, unbalanced (total possible supply 171, total demand 158), non-zero unloading time at each destination, and an integer-valued optimal solution required. The problem is based on real-world data where sources and destinations are located in 25 cities in Poland. The input data – parameters of the problem are presented in Table 1.

Tab. 1

Parameters of the example problem

	Transportation times t_{ij}																
Source/ Destination	D1	D2	D3	D4	D5	D6	D7	D8	D9	D10	D11	D12	D13	D14	D15	D16	Supplies a_i
S1	2	6	5	6	6	8	9	14	11	9	12	14	10	11	13	15	15
S2	4	2	7	5	3	8	5	9	8	7	10	11	11	11	12	13	20
S3	7	1	8	5	3	8	2	6	5	6	8	8	10	9	10	10	23
S4	6	3	5	3	1	5	3	6	4	3	6	7	6	6	7	8	16
S5	7	6	5	2	3	2	6	8	4	2	4	7	3	4	5	8	21
S6	11	7	9	7	6	8	5	3	2	4	4	2	7	7	6	5	17
S7	6	10	2	3	5	1	10	11	7	5	7	9	3	4	6	8	19
S8	14	11	12	9	9	10	8	5	4	6	4	1	9	7	6	3	22
S9	13	10	9	9	8	6	9	8	5	6	2	4	5	3	2	2	18
Demands b_j	12	9	7	15	9	10	12	10	18	9	6	12	6	6	8	9	
Unloading times t'_{ij}	0.33	0.33	0.17	0.5	0.33	0.33	0.33	0.33	0.67	0.33	0.17	0.5	0.17	0.17	0.33	0.33	

Source: [36]

The calculations in [36] were performed for the model with the objective function (9) (transformed) subject to (5) (with = replaced with ≤), (6-7) and (30) in three variants (two of them including additional constraints (8)). The purpose of introducing additional constraints was to "shrink" the feasible set and, in this way, improve the result of calculations. Those additional constraints had nothing to do with considering any actual restrictions of the flow of commodities and were in fact just a "mathematical trick" serving as attempts to improve suboptimal solutions.

- Variant 1 – the problem as described above without additional constraints (8).
- Variant 2 – the problem like in Variant 1 in which additional constraints (8), as described below, were added. Moreover, a new upper bound D, lower than the lowest – "natural" common upper bound $M = 18$ defined in (12), is imposed on the amount of the commodity transported on each route x_{ij}. This upper bound $D = 8$ – the average of the solution of Variant I, i.e., the average of x^*_{ij} $(i = 1, \dots, n;\ j = 1, \dots, m)$ calculated in MATLAB and in Excel. The additional constraints (8) were $x_{ij} \le d_{ij} = D$ $(i = 1, \dots, n;\ j = 1, \dots, m)$.

- Variant 3 – like Variant 2 but the upper bound imposed on variables x_{ij} is decreased to the largest x^*_{ij} less than the upper bound in Variant 2 ($D = 6$).

Calculations whose results are presented in this paper were performed for the model (21) subject to (23) (with = replaced with ≤), (24-25), (27-30), $M = 18$ (a linear QDBTP). A comparison between the results from [36] and the ones calculated for this paper is presented in Table 2.

The parameters t'_{ij} are unloading times only, so $t'_{1j} = t'_{2j} = t'_{3j} = t'_{4j} = t'_{5j} = t'_{6j} = t'_{7j} = t'_{8j} = t'_{9j}$ for each $j = 1,2, \dots ,16$ and there is no need to repeat them in many rows. Unloading times other than 0.5 stand for rounded values of fractions: 0.17 for 1/6, 0.333 for 1/3 and 0.67 for 2/3 what correspond for 10, 20 and 40 minutes, respectively.

Tab. 2

A brief comparison of the results for Example 2

	Solution in [36] – nonlinear problem		Solution calculated for this paper – linear problem
Software	Excel (unknown version) with built-in Solver	MATLAB (unknown version)	Excel 365 with add-in OpenSolver 2.9.4
Calculated minimal delivery time t_{min}	Variant 1:13.3 Variant 2: 9.3 Variant 3: 9.0	Variant 1: 12.72 Variant 2: 12.34 Variant 3: 13.98	6.333
Number of used vehicles/routes for t_{min}	Variant 1: 80 Variant 2: 61 Variant 3: 62	Variant 1: 24 Variant 2: NA Variant 3: NA	33
Total delivery time for t_{min}	Variant 1: 492 Variant 2: 381 Variant 3: 384	Variant 1: 176 Variant 2: NA Variant 3: NA	179

Since Variants 2 and 3 impose upper limits on the quantity of the commodity for each route, then the solutions for the nonlinear model are only fully comparable to the linear model for of Variant 1. Whereas the values t_{min} returned by optimization software theoretically should be equal for the nonlinear model in Variant 1 and the linear model, in fact they were over twice as bad for nonlinear optimization. The above results show that nonlinear optimization can be an inefficient way of solving QDBTP, at least for some of its instances. Calculations in Excel resulted in significant improvements of the value t_{min} when additional upper limits on x_{ij} were added (Variant 2 vs Variant 1: t_{min} better by 4 hours/30.08%, Variant 3 vs Variant 1: t_{min} better by 4.3 hours/32.33%). However, those results are still much worse that $t_{min} = 6.333$ obtained for the linear model. Calculations in MATLAB showed that additional upper limits on x_{ij} improved the result slightly (Variant 2 vs Variant 1: t_{min} better by 0.38 hours/2.99%) or even made it worse (Variant 3 vs Variant 1: t_{min} worse by 1.26 hours/9.91%). Anyway, it is worth noting that the solution for Variant 1 calculated in MATLAB results in lower values for the number of used vehicles/routes and the total delivery time in comparison with any other solution calculated in Excel, no matter if the model was nonlinear or linear.

Dropping the integer constraints imposed on x_{ij} results in a solution with $t_{min} = 6$ in which the variables $x_{ij}$are partially non-integer. The above fact proves the necessity of using integer constraints on x_{ij} in QDBTP.

The data from Table 1 was also used for an instance of the QDTTMTP problem. The minimal total delivery time on all the routes turned out to be 101 hours, the number of used vehicles/routes was 17 and all the deliveries were completed after 12.3333 hours, which is the time of delivering 11 units of the commodity from source 3 to destination 9.

4. DISCUSSION

This paper summarizes the concept of considering delays depending proportionally on the quantity of the transported commodity in time minimization transportation planning. This summary includes extending the existing interpretation of the time-quantity dependence, naming the optimization problems, determining connections with analogical models, and providing an efficient way of calculating the optimal solutions [46].

The main goals of the paper are discussed below.

The first goal was to just formulate extensions of BTP and TTMTP to the case in which the quantity of the commodity causes the increase of the overall delivery time because of loading and unloading the commodity as well as slowing down the trip. Those extensions were named the Quantity Dependent Bottleneck Transportation Problem (QDBTP) and the Quantity Dependent Total Time Minimization Transportation Problem (QDTTMTP), respectively. Whereas QDBTP is not a new extension in the mathematical sense, the possible interpretation of the quantity-time dependence was widened to compare with that presented in the literature. Newly introduced QDTTMTP turned out to be a "time optimization version" of the Fixed Charge Transportation Problem (FCTP).

The second goal was to express QDBTP and QDTTMTP as linear programming problems, based on analogical formulations for BTP and QDBTP. Those formulations bring many advantages.

1. They allow one to find an optimal solution in cases when solving an original nonlinear problem – because of limitations of optimization software – may not.
2. It allows for using general-purpose optimization software for solving, i.e., there is no necessity to create dedicated solving algorithms.

It is necessary to say that the linear formulation of QDBTP (and, to some extent, also of QDTTMTP) results also in some issues that may be perceived negatively. However, the issues are more technical features than real disadvantages. The issues are as follows:

1. In the linear formulation of QDBTP or QDTTMT for each variable x_{ij} standing for the quantity of the transported commodity, an auxiliary y_{ij} binary variable is introduced. It means that twice as many variables are required to compare with the linear formulation. Moreover, the number of linear constraints to compare with the nonlinear formulation increases by $m \cdot n$ for QDTTMTP and by $2m \cdot n$ for QDBTP.
2. In case of QDBTP, on some routes there may be optimal values $y_{ij}^* = 1$ "coupled" with $x_{ij}^* = 0$. This phenomenon means that some routes are incorrectly marked as "used", but it does not affect the optimality of the solution in x_{ij} variables. The only practical consequence is that the total number of used routes/vehicles must be calculated as the number of x_{ij}^* for which $x_{ij}^* > 0$, (sgn x_{ij}^*) not the number of y_{ij}^* for which $y_{ij}^* = 1$.

The third goal was to discuss the existence of the "guaranteed" integer optimal solutions in the optimization models under consideration. It turned out that, unlike in case of BTP, QDBTP does not have "guaranteed" integer optimal solutions (in x_{ij} variables) even if all the maximal supplies of sources and the demands of destinations are integer. Technically speaking, it means that whenever the amount of the commodity on each route must be an integer, imposing the integer constraints on variables x_{ij} is necessary. This feature may be a disadvantage only in cases when the abovementioned integer constraints are the reason to slow down the calculations unacceptably.

5. CONCLUSIONS

In this paper, it was shown that QDBTP formulated as a linear programming problem can be an efficient approach to time-critical transportation planning. The most important advantage is that this approach results in obtaining optimal solutions in cases in which solving original nonlinear problems failed. Moreover, problems of that kind can be solved optimally using general purpose optimization software without the necessity of creating dedicated algorithms and software that implement them. The abovementioned fact does not exclude, however, searching for dedicated algorithms if using standard linear programming turns out to result in an unacceptably long time to solve the problems essential for real-world applications. The linear formulation of QDBTP also allows for easily formulating problems with second-level optimality criteria regarding many possible economic and ecological aspects of transportation planning.

QDTTMTP - an extension of TTMTP with an analogical quantity-dependent component of the objective function as well as its linear formulation was also introduced in the study.

A possible occurrence of multiple optimal solutions in QDBTP makes it possible to consider second-level optimality criteria for QDBTP. Namely, for a given t_{min} (the minimal delivery time – the optimal value of the objective function of QDBTP) there can be alternative solutions x_{ij}^* which differ in features such as like the total delivery time, number of used vehicles/routes, delivery cost, fuel consumption, mileage. Each of those features (as well as possibly some other) can become a second-level optimality criterion, and a new optimization problem can be formulated. Such a problem is of the form "minimize a second-level optimality criterion (e.g., the total delivery cost) subject to the constraints of the initial problem completed the delivery time set to t_{min}. Whereas the idea of second-level optimality criteria for QDBTP is quite simple, it requires a separate paper to be presented precisely.

Both QDBTP and QDTTMTP can obviously be developed further in many aspects. Some propositions are mentioned below. The two models can easily be extended to include issues of packing and storage of the transported commodity or the possibility of using multiple vehicles on a single route.

New issues may arise if we consider using electric-powered vehicles (EV). Time minimization in transportation planning assumes implicitly travelling with the maximal speed and acceleration allowed by the traffic law and technical-safety requirements. However, the speed and, especially, acceleration strongly affect consumption of energy stored in batteries of the vehicles [44]. Moreover, the amount of energy required to accelerate the vehicle increases with the mass of the transported cargo. Even if we take into account energy recuperation during braking, there is still a possibility that at some weight of the carried cargo, a "breaking point" can be attained at which an extra recharging of the batteries during the trip is necessary. Because recharging the batteries in EVs usually takes much longer compared with refueling the comparable internal combustion vehicles, the total delivery time may increase dramatically. The above considerations show that there is a need to develop QDBTP and QDTTMTP versions for EVs. They must reflect the fact that, when the weight of the cargo exceeds some level, then there is a necessity of changing the driving style to a slower, but more energy-saving one, or, if it is impossible to avoid, to add the recharging time to the total trip time.

Finally, reliable modeling the delivery plans may not be restricted to deterministic cases only. It may also require taking into account various aspects of possible uncertainty. For example, an important factor can be the random availability of the real-world fleets of vehicles. A promising approach seems to be using the readiness of each vehicle described by semi-Markov reliability models [45]. A draft idea is to combine the parameters of the deterministic QDBTP and QDTTMTP models with parameters of the models in [45] in order to obtain tighter, readiness-constrained capacity bounds. The above idea is intended to be the base for a possible further development of non-deterministic extensions of the time-minimizing delivery models.

At the same time, the study presents certain limitations as follows.

- While the paper highlights the advantage of formulating QDBTP as a linear programming problem (LP) to overcome the challenges of the original nonlinear formulation, it also explicitly acknowledges a crucial limitation: the potential for unacceptably long addressing times when employing general-purpose LP solvers for problems of a scale relevant to real-world applications. Linear programming can still face significant computational burdens with a large number of variables and constraints. Real-world transportation problems often involve numerous origins, destinations, commodities, vehicle capacities, and time windows. The linear formulation of QDBTP, while enabling optimality guarantees, might generate a problem instance with a size that overwhelms standard LP solvers, leading to impractical solution times for time-sensitive planning scenarios. This limitation suggests that the practical applicability of the proposed LP formulation might be restricted to smaller-scale problems or might necessitate the use of highly optimized commercial LP solvers.
- The research conducted introduces the valuable concept of leveraging the potential for multiple optimal solutions in QDBTP to optimize based on secondary criteria once the minimal delivery time is achieved. However, this remains a conceptual introduction without a concrete formulation or detailed analysis. The paper states that "the idea of second-level optimality criteria for QDBTP is quite simple; it requires a separate paper to be presented precisely." This clearly indicates that the actual mathematical formulation of how to incorporate and optimize these secondary objectives within the constraints of the primary (minimal delivery time) problem is not developed within the current study. Questions remain about how to effectively model these criteria, how to handle potential trade-offs between different secondary objectives, and the computational implications of addressing such second-level optimization problems. This limitation entails that the paper does not provide the methodological framework or computational validation for implementing these second-level optimizations.

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